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THE ORBITAL AND SPIN PARTS OF ANGULAR MOMENTUM OF LIGHT

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ABSTRACT

In this paper we have shown that the separation of the total angular momentum of the electromagnetic field into its orbital and spin parts. It is dictated by quantum mechanics of photons reproduces. Therefore, the results are derived from the properties of Maxwell fields by Darwin, with the correspondence results that derived heuristically by many authors.

KEYWORDS: angular momentum of light, quantum mechanics of photons, Riemann-Silberstien vector.

INTRODUCTION

Darwin's theory of evolution is the widely held notion that all life is related and has descended from a common ancestor. Darwin's general theory presumes the development of life from non-life and stresses a purely naturalistic. That is complex creatures evolve from more simplistic ancestors naturally over time. In this work, we will review the work done by Iwo Bialynicki-Birula¹ and Zofia Bialynicka-Birula² [1], to show that the Darwin separation of the total angular momentum for an arbitrary electromagnetic field into two parts follows from the photon picture of the electromagnetic field with some remarks.

GENERATOR OF THE POINCARÉ GROUP:

The Poincaré group is the group of Minkowski spacetime isometries. It is a ten dimensional noncompact Lie group. The abelian group of translations is a normal subgroup, while the Lorentz group is also a subgroup, the stabilizer of the origin. The Poincaré group itself is the minimal subgroup of the affine group which includes all translations and Lorentz translations. More precisely, it is a semi direct product of the translations and Lorentz group.

In a relativistic theory, we must first of all define the operators representing ten generators of the Poincaré group, that must obey the following commutation relations

$$[\hat{H}, \hat{P}_i] = 0, [\hat{H}, \hat{J}_i] = 0, [\hat{H}, \hat{K}_i] = -ihc\hat{P}_i \quad (1)$$

$$[\hat{P}_i, \hat{P}_j] = 0, [\hat{J}_i, \hat{J}_j] = ih\epsilon_{ijk}\hat{J}_k, [\hat{K}_i, \hat{K}_j] = -ihc^2\epsilon_{ijk}\hat{J}_k \quad (2)$$

$$[\hat{J}_i, \hat{P}_j] = ih\epsilon_{ijk}\hat{P}_k, [\hat{J}_i, \hat{K}_j] = ih\epsilon_{ijk}\hat{K}_k, [\hat{K}_i, \hat{P}_j] = ih\delta_{ij}\hat{H} \quad (3)$$

Where, \hat{P} is the generators of translation in space (momentum), \hat{H} is the translation in time (energy), \hat{J} is the rotation (angular momentum), \hat{K} are the Lorentz boosts (momentum of energy), and C is speed of light.

The representation of the generators of the Lorentz group for massless particles was given by Lomont and Mose [2]. We will review here a modified version of these generators for photons [3], [4], [6] that exhibits its geometrical meaning. The momentum operator, by definition, acts on the wavefunctions in momentum representation as a multiplication by hk . The complete list of generators also contains the operator of angular momentum and the boost operator [1]

$$\hat{H} = hw_k \quad (4)$$

$$\hat{P} = hk \quad (5)$$

$$\hat{J} = i\hbar D \times k + \hbar \hat{x} n_k \tag{6}$$

$$\hat{K} = i\hbar w_k D \tag{7}$$

Where $n_k = \frac{k}{|k|}$, the photon helicity operator \hat{x} has the eigenvalues ± 1

$$D = \nabla_k - i\hat{x} \alpha(k) \quad , \quad \nabla_k = \frac{\partial}{\partial k} \tag{8}$$

stands for the covariant derivative on the light cone. These operators act on the two-component photon wavefunctions

$$f(k) = \begin{pmatrix} f_L(k) \\ f_R(k) \end{pmatrix} \tag{9}$$

and satisfy the commutation relations (1), (2) and (3) appropriate for the Poincaré group. The components of the photon wavefunction correspond to two eigenvalues of \hat{x} ,

$$\hat{x} \begin{pmatrix} f_L(k) \\ f_R(k) \end{pmatrix} = \begin{pmatrix} f_L(k) \\ -f_R(k) \end{pmatrix} \tag{10}$$

here L and R are used to denote the eigenfunctions of the helicity operator since they correspond to left-handed and right-handed circular polarization.

The properties of the covariant derivative are obtained from the commutation relations for the angular momentum and they read [1],

$$[D_i, D_j] = i\hat{x} \epsilon_{ijl} \frac{n_l}{|k|^2} \tag{11}$$

These conditions determine the vector $\alpha(k)$ up to a gauge transformation

$$\alpha(k) \rightarrow \alpha(k) + \nabla_k \varphi(k) \tag{12}$$

Which is connected to the change of the phase of the wavefunction, in analogy to the theory of charged particles coupled to an electromagnetic field. In order to solve the problem of the total angular momentum separation into two parts for the classical electromagnetic field, we shall employ the correspondence between the fundamental physical quantities (energy, momentum, and angular momentum) in photon quantum mechanics and in Maxwell theory. In the quantum mechanics of photons these quantities are represented by the operators (4)– (7). The generators (4), (5), (6) and (7) are Hermitian with respect to the following Lorentz-invariant scalar product [1],

$$\langle f | g \rangle = \int \frac{d^3k}{\hbar w_k} f^\dagger(k) \cdot g(k) = \int \frac{d^3k}{\hbar w_k} [f_L^*(k) g_L(k) + f_R^*(k) g_R(k)] \tag{13}$$

MAXWELL'S THEORY

Maxwell's equations can be cast into covariant form. The Einstein expression of it, is that, the general laws of nature are to be expressed by equations which holds for all systems of coordinates that are covariant with respect to any substitution whatever generally covariant.

Maxwell's theory of electromagnetism is alongside with Einstein's theory of gravitation, on the most classical field theories. The revolutionary work of Maxwell, published in 1865 took the individual and seemingly unconnected phenomena of electricity and magnetism and brought them into a Coherent and unified theory [16],[18]. This unified theory of electricity and magnetism depicts the behavior of two fields. Maxwell discussed his idea in terms of model in which vacuum was like an elastic solid, he tried to explain the meaning of his new equations in term of mathematical model. There was much reluctance to accept his theory, first because of the model, and second because there was at first no experimental justification. But today, we understand better that what counts are the equations themselves and not the model used to get them. If we chose units in which $\mu_0 = \epsilon_0 = c = 1$, then the covariant form of Maxwell's equations take the form[7], [8]:

$$\nabla \cdot E = \rho \tag{14}$$

$$\nabla \times B - \frac{\partial E}{\partial t} = J \quad (15)$$

$$\nabla \cdot B = 0 \quad (16)$$

$$\nabla \times E + \frac{\partial B}{\partial t} = 0 \quad (17)$$

Where E is the electric, and B is magnetic field, ρ is the charge density, and J is the current density. Taking the divergence of equation (14) and substituting equation (15) into resulting equation. Now, we can obtain the continuity equation [22], [23]

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0 \quad (18)$$

Now we have used the fact that for any vector H and scalar ψ the following identities hold:

$$\nabla \cdot (\nabla \times H) = 0 \quad (19)$$

$$\nabla \times (\nabla \psi) = 0 \quad (20)$$

Also, since equation (16) always holds, this means that B must be curl of a vector function, namely the vector potential A ,

$$B = \nabla \times A \quad (21)$$

Substituting equation (21) into equation (17) we obtain

$$\nabla \times \left(E + \frac{\partial A}{\partial t} \right) = 0 \quad (22)$$

Which means that the quantity with vanishing curl in equation (22) can be written as the gradient of scalar function, namely the scalar potential ϕ .

$$E = -\nabla \phi - \frac{\partial A}{\partial t} \quad (23)$$

The minus sign attached to the gradient is for technical convenience. These quantities, in Maxwell theory are given as space integrals of corresponding densities built from quadratic expressions in field vectors. The convenient tool in this construction is a complex vector F , that was called the Riemann–Silberstein (RS) vector in [1],[5], [9]. and given by

$$F = \sqrt{\frac{\epsilon_0}{2}} (E + icB) \quad (24)$$

The Maxwell equations expressed in terms of F are:

$$\partial_t F(r, t) = -ic \nabla \times F(r, t), \quad \nabla \cdot F(r, t) = 0, \quad (25)$$

Now all of the field energy H , the field momentum P , the field angular momentum J , and the field moment of energy K can be constructed from the energy–momentum tensor of the electromagnetic field, and they expressed in terms of the RS vector as follows [1],

$$H = \frac{1}{2} \int d^3 r \left[\epsilon_0 E^2 + \frac{B^2}{\mu_0} \right] = \int d^3 r F^* \cdot F \quad (26)$$

$$P = \int d^3 r [\epsilon_0 E \times B] = \frac{1}{2i} \int d^3 r F^* \times F \quad (27)$$

$$J = \int d^3r r \times [\epsilon_0 E(r) \times B(r)] = \frac{1}{2i} \int d^3r r \times (F^* \times F) \quad (28)$$

$$K = \frac{1}{2} \int d^3r r \left[\epsilon_0 E^2 + \frac{B^2}{\mu_0} \right] = \int d^3r r (F^* \cdot F) \quad (29)$$

The above quantities, like their counterparts in photon quantum mechanics (4),(5), (6), and (7) serve as the generators of Poincaré transformations of the electromagnetic field. . They have analogous algebraic properties of the Poincaré group (1),(2) and (3), with quantum commutators replaced by Poisson brackets, $\frac{[a,b]}{i\hbar} \rightarrow [a, b]$.

The solutions of Maxwell equations in vacuum can be decomposed into plane waves with positive and negative frequencies. This decomposition gives the following Fourier representation of $F(r, t)$:

$$\int F(r, t) = \sqrt{N} \int \frac{d^3k}{(2\pi)^{3/2}} e(k) [f_L(k) e^{-i\omega_k t + ik \cdot r} + f_R^*(k) e^{i\omega_k t - ik \cdot r}] \quad (30)$$

Where the complex polarization vector

$$e(k) = \frac{1}{\sqrt{2}} [L_1(k) + iL_2(k)] \quad (31)$$

has the following properties:

$$ck \times e(k) = -i\omega_k e(k) \quad (32)$$

$$e(k) \cdot e(k) = 0 \quad (33)$$

$$e^*(k) \cdot e(k) = 1 \quad (34)$$

$$e^*(k) \times e(k) = in_k \quad (35)$$

$$e^*(k) \cdot e(-k) = 0 \quad (36)$$

$$e(k) \times e(k) = 0 \quad (37)$$

$$e_i^*(k) e_j(k) = \frac{1}{2} \left(\delta_{ij} + i\epsilon_{ijl} \frac{k_l}{|k|} \right) \quad (38)$$

SEPARATION OF ANGULAR MOMENTUM

In the following we are going to justify the identification of the Fourier coefficients with the components of the photon wave function in the formula (30) by unifying the field picture and the photon picture. We can see that the second term in the left hand side of (30) involves complex conjugation, and this is dictated by the fact that the photon energy is always positive. Therefore, the time evolution of the wavefunction is given by the factor $e^{-i\omega_k t}$ Therefore, the reversal of the sign in the exponent requires complex conjugation and then we discuss the pulling out of the factor \sqrt{N} to assure the normalization of f .

Firstly, we shall combine now the field picture and the photon picture to obtain the decomposition of the total angular momentum of the field. To this end, we use the substituting of the Fourier representation of the field into the formulas (26), (27),(28) and (29),

$$H = N \int \frac{d^3k}{h\omega_k} f^\dagger(k) \cdot h\omega_k f(k) \quad (39)$$

$$P = N \int \frac{d^3k}{h\omega_k} f^\dagger(k) \cdot hkf(k) \quad (40)$$

$$J = N \int \frac{d^3k}{hw_k} f^\dagger(k) \cdot [ihD \times k + h\hat{x}n_k]f(k) \quad (41)$$

$$K = N \int \frac{d^3k}{hw_k} f^\dagger(k) \cdot ihw_k Df(k) \quad (42)$$

We have to note that, the resulting expressions have the form of quantum mechanical expectation values:

$$H = N \langle f | \hat{H} | f \rangle \quad (43)$$

$$P = N \langle f | \hat{P} | f \rangle \quad (44)$$

$$J = N \langle f | \hat{J} | f \rangle \quad (45)$$

$$K = N \langle f | \hat{K} | f \rangle \quad (46)$$

As Darwin had anticipated, we can see that, these formulas exhibit a perfect agreement between the results obtained from the particle picture and from the field picture. Also, every value calculated for the total electromagnetic field is a product of the quantum mechanical average value per one photon, multiplied by N [1]. This means that, the normalization factor N is the total number of photons [1]. Now, we may unambiguously split that the total angular momentum of the electromagnetic field (41) into two parts as done in [1]. The vector J_0 whose integrand is perpendicular to the wavevector is the orbital part and the vector J_s whose integrand is parallel to the wavevector is the spin part represented by helicity:

$$J_0 = N \int \frac{d^3k}{hw_k} f^\dagger(k) \cdot [ihD \times k]f(k) \quad (47)$$

$$J_s = N \int \frac{d^3k}{hw_k} f^\dagger(k) \cdot h\hat{x}n_k f(k) = N \int \frac{d^3k}{w_k} n_k [|f_L(k)|^2 - |f_R(k)|^2] \quad (48)$$

The final step of our studding of the above analysis is the proof that the expressions for J_0 and J_s coincide with those obtained by Darwin. In [1] they employ the relation between $E(k)$ and $f(k)$ that follows from the formula(30):

$$E(k) = \sqrt{\frac{N}{2 \epsilon_0}} [e(k)f_L(k) + e^*(k)f_R(k)] \quad (49)$$

Where $E(k)$ is the plane -wave component of the electric field,

$$E(r, t) = \int \frac{d^3k}{(2\pi)^{3/2}} [E(k)e^{-i\omega_k t + ik \cdot r} + c] \quad (50)$$

Now, by using the properties of the polarization vectors (35) and (37) we get [1]:

$$\begin{aligned} & -2i\epsilon_0 \int \frac{d^3k}{c|k|} E^*(k) \times E(k) \\ & = -iN \int \frac{d^3k}{c|k|} [e^*(k) \times e(k)|f_L(k)|^2 + e(k) \times e^*(k)|f_R(k)|^2] = J_s \quad (51) \end{aligned}$$

Now, we have to note that, the separation of the total angular momentum into its orbital and spin parts is conserved in time since both parts are separately time independent.

ANOTHER SEPARATION OF ANGULAR MOMENTUM

Now by using the formula (13) we may again unambiguously split that the total angular momentum of electromagnetic field (41) into two parts. The vector J_{OM} whose integrand is perpendicular to the wave vector is the orbital part and the vector J_{SM} whose integrand is parallel to the wave vector is the spin part represented by helicity ; then we can rewrite (41) by using (13) in the form:

$$J = N \int \frac{d^3k}{hw_k} [f_L^*(k)f_L(k) + f_R^*(k)f_R(k)]. [ihD \times k + h\hat{x}n_k] \quad (52)$$

There for:

$$J_{OM} = N \int \frac{d^3k}{hw_k} [f_L^*(k)f_L(k) + f_R^*(k)f_R(k)]. [ihD \times k] \quad (53)$$

$$J_{SM} = N \int \frac{d^3k}{w_k} [f_L^*(k)f_L(k) + f_R^*(k)f_R(k)]. [\hat{x}n_k] \quad (54)$$

We can see that, the formulas (53) and (54) in our above analysis are coincide authors, and are separately time independent.

CONCLUSIONS

As a conclusion, in this paper we discussed the total angular momentum of electromagnetic field into its two parts, the orbital and spin. Our main tools, is the quantum mechanics of photons. In fact we revisit the results obtained by Darwin using Maxwell fields properties. We have also shown that our results coincide with previous results obtained vi several authors. Our last observation was that, when comparing the energy momentum and the total angular momentum of electromagnetic field, the two parts of momentum can't be expressed as an integral forms of local densities, and the formulas (53) and (54) in our above analysis are coincide authors, and are separately time independent.

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